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ON THE SYMBOLIC CALCULUS IN HOMOGENEOUS BANACH ALGEBRAS

BY

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ABSTRACT

We construct a strongly homogeneous Banach algebra B so that for an appropriate positive integer n, (1) $A(\mathbf{T}) \subsetneqq B \subsetneqq C(\mathbf{T})$; (2) n-times continuously differentiable functions operate on B.

Let $A(\mathbf{T})$ denote the algebra of absolutely convergent Fourier series, and let $C(\mathbf{T})$ denote the space of continuous functions on the circle group $\mathbf{T}$. In [5], the author constructed a homogeneous Banach algebra B so that (1) $A(\mathbf{T}) \subsetneqq B \subsetneqq C(\mathbf{T})$, and (2) non-analytic functions operate on B. In this note, we continue our study of these algebras, and obtain a more precise symbolic calculus for them. We will show that n-times continuously differentiable functions operate on the Banach algebra B, for an appropriate integer n. The method of proof will illustrate a relationship between B and the real interpolation spaces of Lions and Peetre (see [1] and [3]). Our principal results are Theorems 2.2 and 2.10. We begin with some notations and comments.

1. Let B be a commutative, semi-simple, self-adjoint Banach algebra with maximal ideal space $\mathbf{T}$. We view B as an algebra of continuous functions on $\mathbf{T}$. B will be called homogeneous provided the following two properties hold:

(1) For every $a \in \mathbf{T}$, the mapping $f(x) \to f(x+a)$ is an isometry of B into itself.

(2) For every $f \in B$, we have $\lim_{a\to 0} \| f(x+a) - f(x) \|_B = 0$.

B will be called strongly homogeneous provided we also have

(3) For every integer k, the operator $f(x) \to f(kx)$ maps B into itself and is of norm 1.

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In [5], we obtained a negative solution to the "dichotomy problem" in the context of homogeneous Banach algebras. Specifically, we proved:

THEOREM 1.1. *There exists a strongly homogeneous Banach algebra B satisfying the following two properties*:

(a) $A(\mathbf{T}) \subsetneqq B \subsetneqq C(\mathbf{T})$.

(b) $\sup_{\substack{\psi \in B \\ \psi \text{ real} \\ \|\psi\|_B \leqq 1}} \|e^{ir\psi}\|_B \leqq 6|r|^{3/2} e^{5|r|^{1/2}}$ *for* $|r| \geqq 1$.

In particular, non-analytic functions operate on B.

The subexponential growth of

$$N_B(r) = \sup_{\substack{\psi \in B \\ \psi \text{ real} \\ \|\psi\|_B \leqq 1}} \|e^{ir\psi}\|_B$$

given in estimate 1.1(b) implies only that certain classes of infinitely differentiable functions operate on B (see [4], chapter 6, section 7). Following a suggestion of Y. Katznelson, we seek a more precise symbolic calculus for homogeneous Banach algebras in general, and for the algebra B of 1.1 in particular. In this note, we will show that n-times continuously differentiable functions operate on B. Here n is an appropriate positive integer. A necessary tool in our arguments will be the "espaces de moyenne" of Lions and Peetre (see [1], chapter 3; and [3]). Let us recall some basic facts concerning these spaces.

Let X be a Banach space. We denote by $L^*_p(X)$ the space of all functions $f: (0,\infty) \to X$ which are strongly measurable with respect to the measure dt/t, and for which

$$\|f\|_{L^*_p(X)} = \begin{cases} \left(\int_0^\infty \|f(t)\|_X^p \frac{dt}{t} \right)^{1/p} < \infty & \text{if } 1 \leqq p < \infty, \\ \operatorname*{ess\,sup}_{t>0} \|f(t)\|_X < \infty & \text{if } p = \infty. \end{cases}$$

In case $X = \mathbf{C}$, the complex numbers, we write L^*_p in place of $L^*_p(X)$.

Throughout the remainder of this section, B^0 and B^1 will denote Banach spaces continuously embedded in a topological linear space. Define the familiar spaces $B^0 + B^1$ and $B^0 \cap B^1$ as in [1], section 3.2 We introduce the function norm

$$J(t,x) = \max(\|x\|_{B^0}, t\|x\|_{B^1})$$

for $0 < t < \infty$ and $x \in B^0 \cap B^1$. Also, we define for all $x \in B^0 + B^1$, and $0 < t < \infty$,

$$K(t,x)=\inf(\|x_0\|_{B^0}+t\|x_1\|_{B^1}),$$

the infimum taken over all representations $x = x_0 + x_1$ with $x_i \in B^i$, $i = 0, 1$.

Let $0<\theta<1$ and $1 \leqq q \leqq \infty$. We define $(B^0, B^1)_{\theta,q,J}$ (or $B_{\theta,q,J}$) to be the space of all elements $x \in B^0+B^1$ for which there exists a strongly measurable function $u:(0,\infty)\to B^0\cap B^1$ so that

(1) $x = \int_0^\infty u(t)\frac{dt}{t} \qquad (u \in L_1^*(B^0+B^1))$

and

(2) $t^{-\theta}J(t,u(t))\in L_q^*$.

Then $(B^0, B^1)_{\theta,q,J}$ becomes a Banach space under the norm

$$\|x\|_{\theta,q,J}=\inf\|t^{-\theta}J(t,u(t))\|_{L_q^*},$$

the infimum taken over all functions u satisfying conditions (1) and (2) above.

We next turn to the intermediate spaces generated by the "K-method". If $0<\theta<1$ and $1\leqq q\leqq\infty$, we define $(B^0,B^1)_{\theta,q,K}$ (or $B_{\theta,q,K}$) to be the space of all elements $x\in B^0+B^1$ so that

$$\|x\|_{\theta,q,K}=\|t^{-\theta}K(t,x)\|_{L_q^*}<\infty.$$

Under the norm $\|\cdot\|_{\theta,q,K}$, $(B^0,B^1)_{\theta,q,K}$ becomes a Banach space.

We refer the reader to [1], chapter 3, and [3] for the basic properties of these spaces. In particular, the following results will be required in the sequel.

THEOREM 1.2. *Let* $0<\theta<1$ *and* $1\leqq q\leqq\infty$. *Then* $(B^0,B^1)_{\theta,q,J}=(B^0,B^1)_{\theta,q,K}$. *Moreover, for all* $x\in(B^0,B^1)_{\theta,q,K}$, *we have*

(a) $\|x\|_{\theta,q,J}\leqq 4e\|x\|_{\theta,q,K}$,

and

(b) $\|x\|_{\theta,q,K}\leqq\left(\frac{1}{1-\theta}+\frac{1}{\theta}\right)\|x\|_{\theta,q,J}$.

The proof of this theorem may be found in section 3.2.3 of [1]. We will actually make use of the constants arising in parts (a) and (b) above.

THEOREM 1.3. *Let* $0<\alpha<\theta<1$, *and let* $1\leqq q\leqq\infty$. *Then, for all* $x\in B^0\cap B^1$, *we have*

$$\|x\|_{\theta,q,K}\leqq\frac{C}{(1-\alpha)^{1/q}}\|x\|_{(B_{\alpha,1,J},B^1)_{s,q,K}},$$

where $(1-s)\alpha+s\cdot 1=\theta$. *The constant* C *is independent of* α, θ, q *and* x.

This result is a very special case of the well-known theorem of reiteration (see [1], section 3.2.4). We have only stated the theorem in the form required for our theory. Again, the constants arising will be of crucial importance.

THEOREM 1.4. *Let $0 < \theta < 1$, and let $1 \leqq q \leqq \infty$. Then*

$$\| x \|_{\theta, q, J} \leqq C \| x \|_{B^0}^{1-\theta} \| x \|_{B^1}^{\theta},$$

for all $x \in B^0 \cap B^1$. The constant C is independent of x, θ, and q.

This result follows easily by the methods of [3], chapter 1, section 3.

2. In this section, we will obtain our principal results concerning the symbolic calculus of homogeneous Banach algebras. To this end, we recall the constructions introduced in [5].

DEFINITION 2.1. Let B^0 and B^1 be Banach spaces so that $B^0 \subseteq B^1$, with continuous inclusion, and let B^0 be dense in B^1. Let $f: [0,\infty) \to [1,\infty)$ be a continuous, non-decreasing function so that $f(t) \to \infty$ as $t \to \infty$, and $f(t)/t \to 0$ as $t \to \infty$. For $x \in B^0$, we define an admissible representation for x to be an expansion of x in the form $x = \sum_{k=1}^{n} a_k x_k$, with $x_k \in B^0$ and $\| x_k \|_{B^1} \leqq 1$, $1 \leqq k \leqq n$. Define

$$\| x \|_f = \inf \sum | a_k | f(\| x_k \|_{B^0}),$$

the infimum taken over all admissible representations $x = \sum a_k x_k$. Define $(B^0, B^1)_f$ (or B_f) to be the completion of B^0 under the norm $\| \cdot \|_f$.

In [5], section 2, it is shown that $\| x \|_{B^1} \leqq \| x \|_f \leqq C \| x \|_{B^0}$, for all $x \in B^0$. Moreover, if the norms of $(B^0)^*$ and $(B^1)^*$ are not equivalent, then the proper inclusions $B^0 \subsetneqq B_f \subsetneqq B^1$ obtain. Finally, if f satisfies the additional property $f(t_1 t_2) \leqq f(t_1) f(t_2)$, for all $t_1, t_2 \geqq 0$, and if B^0 and B^1 are Banach algebras whose multiplications coincide, then $(B^0, B^1)_f$ is a Banach algebra. The case $B^0 = A(\mathbf{T})$, $B^1 = C(\mathbf{T})$ and $f_0(t) = \log(t + e^2)$ is studied in detail in [5]. There it is shown that non-analytic functions operate on the strongly homogeneous Banach algebra $(A(\mathbf{T}), C(\mathbf{T}))_{f_0}$. We will continue our study of this algebra in the sequel, and obtain the following theorem. (C^n will denote the space of 2π periodic, n-times continuously differentiable functions on $\mathbf{R}$).

THEOREM 2.2. *Let $f_0(t) = \log(t + e^2)$, and let $B = (A(\mathbf{T}), C(\mathbf{T}))_{f_0}$. Then*

$$\sup_{\substack{\psi \in B \\ \psi \text{ real} \\ \|\psi\|_B \leqq 1}} \| e^{ir\psi} \|_B \leqq C | r |^5,$$

for $|r| \geqq 1$. *Here* C *is an absolute constant. In particular,* C^6 *functions operate on* B.

The reader may wish to compare this result with Theorem 1.1. Our proof of 2.2 will require a comparison of B with the real interpolation spaces $(A(\mathbf{T}), C(\mathbf{T}))_{\theta,1,J}$, $0 < \theta < 1$.

NOTATION 2.3. Throughout the remainder of this paper, C will denote a constant independent of all parameters which may appear. Its value may change from place to place. Let $f_\alpha(t) = (t + e^2)^\alpha$ for $0 < \alpha < 1$, $t \geqq 0$. We write $B_{(\alpha)}$ in place of $(B^0, B^1)_{f_\alpha}$, and let $\|\cdot\|_{(\alpha)}$ denote the norm of this space.

LEMMA 2.4. *Assume the notations of* 2.1 *and* 2.3, *and let* $0 < \theta < 1$. *Then*

$$\|x\|_{(1-\theta)} \leqq C\|x\|_{B^0}^{1-\theta}\|x\|_{B^1}^{\theta},$$

for all $x \in B^0$.

PROOF. Since $B^0 \subseteq B^1$ with continuous inclusion, we may choose $C_0 > 0$ so that

$$\|x\|_{B^1} \leqq C_0\|x\|_{B^0}, \tag{1}$$

for every $x \in B^0$. Moreover, if $x \in B^0$, with $x \neq 0$, we have $x = \|x\|_{B^1}(x/\|x\|_{B^1})$ is an admissible representation for x. Therefore,

$$\begin{aligned}\|x\|_{(1-\theta)} &\leqq \|x\|_{B^1}(\|x\|_{B^0}/\|x\|_{B^1} + e^2)^{1-\theta} \\ &\leqq \|x\|_{B^1}^{\theta}(\|x\|_{B^0} + e^2\|x\|_{B^1})^{1-\theta} \\ &\leqq (1 + C_0e^2)\|x\|_{B^0}^{1-\theta}\|x\|_{B^1}^{\theta},\end{aligned}$$

the last inequality following by (1). This completes the proof.

LEMMA 2.5. *Assume the notations of* 2.1 *and* 2.3, *and let* $0 < \theta < 1$. *Then* $(B^0, B^1)_{\theta,1,J} = B_{(1-\theta)}$. *Moreover, there exist constants* C_1 *and* C_2 (*independent of* θ) *so that*

$$C_1\|x\|_{\theta,1,J} \leqq \|x\|_{(1-\theta)} \leqq C_2\|x\|_{\theta,1,J}, \tag{*}$$

for all $x \in B^0$.

PROOF. Let $x \in B^0$, and let $x = \sum a_k x_k$ be an admissible representation for x. Then

$$\|x\|_{\theta,1,J} \leqq \sum |a_k|\,\|x_k\|_{\theta,1,J} \leqq C\sum |a_k|\,(\|x_k\|_{B^0} + e^2)^{1-\theta}$$

by Theorem 1.4. This implies the first inequality of (*). The second inequality follows by Lemma 2.4 and Proposition 1.1, chapter 4 of [3].

We state the following simple lemma without proof.

LEMMA 2.6. *Let* $0 < \sigma < 1$. *Then for all* $t \geqq 0$, *we have*

$$\log(t + e^2) \leqq \frac{1}{\sigma}(t + e^2)^{\sigma}.$$

LEMMA 2.7. *Assume the notations of* 2.1 *and* 2.3, *define* $f_0(t) = \log(t + e^2)$, *and let* $B = (B^0, B^1)_{f_0}$. *Let* $0 < \alpha < 1$, *and suppose* $x \in B^0$ *with* $\|x\|_{B^1} \leqq 1$. *Then*

$$\|x\|_B \leqq \frac{C}{(1-\alpha)^2}[\log(\|x\|_{(1-\alpha)} + e^2)]^2.$$

PROOF. Observe first that for $0 < \theta < 1$, we have

$$\|x\|_B \leqq \frac{1}{1-\theta}\|x\|_{(1-\theta)}, \tag{1}$$

for all $x \in B^0$.

To establish (1), let $x = \Sigma a_k x_k$ be an admissible representation for x. Then by Lemma 2.6,

$$\|x\|_B \leqq \sum |a_k| \log(\|x_k\|_{B^0} + e^2) \leqq \frac{1}{1-\theta}\sum |a_k|(\|x_k\|_{B^0} + e^2)^{1-\theta}. \tag{2}$$

Clearly (2) implies (1).

Now let $x \in B^0$, with $\|x\|_{B^1} \leqq 1$, and define $\theta \in (\alpha, 1)$ by

$$1 - \theta = \frac{(1-\alpha)}{\log(\|x\|_{(1-\alpha)} + e^2)}.$$

Choose $0 < s < 1$ so that $\theta = (1-s)\alpha + s \cdot 1$.

By (1) and Lemma 2.5, we see

$$\|x\|_B \leqq \frac{C}{1-\theta}\|x\|_{\theta,1,J}. \tag{3}$$

Using Theorems 1.2 and 1.3, we then obtain

$$\begin{aligned} \|x\|_B &\leqq \frac{C}{1-\theta}\|x\|_{\theta,1,K} \\ &\leqq \frac{C}{(1-\theta)(1-\alpha)}\|x\|_{(B_{\alpha,1,J}, B^1)_{s,1,K}} \qquad (4)\\ &\leqq \frac{C}{(1-\theta)(1-\alpha)}\left(\frac{1}{s} + \frac{1}{1-s}\right)\|x\|_{(B_{\alpha,1,J}, B^1)_{s,1,J}}. \end{aligned}$$

Since $s = (\theta - \alpha)/(1-\alpha)$, inequality (4), Theorem 1.4, and the fact that $\|x\|_{B'} \leqq 1$ yield

$$\|x\|_B \leqq \frac{C}{1-\theta}\left(\frac{1}{\theta-\alpha} + \frac{1}{1-\theta}\right)\|x\|^{1-s}_{\alpha,1,J} \tag{5}$$
$$\leqq \frac{C}{1-\theta}\left(\frac{1}{\theta-\alpha} + \frac{1}{1-\theta}\right)\|x\|^{1-s}_{(1-\alpha)},$$

the last estimate following by Lemma 2.5.

Since

$$1 - s = \frac{1}{\log(\|x\|_{(1-\alpha)} + e^2)} \quad \text{and} \quad 1-\theta = \frac{(1-\alpha)}{\log(\|x\|_{(1-\alpha)} + e^2)},$$

a simple numerical calculation in (5) gives

$$\|x\|_B \leqq \frac{C}{(1-\alpha)^2}[\log(\|x\|_{(1-\alpha)} + e^2)]^2,$$

as claimed.

LEMMA 2.8. *Let* $f_0(t) = \log(t + e^2)$, *and define* $B = (A(\mathbf{T}), C(\mathbf{T}))_{f_0}$. *Let* $\psi \in A(\mathbf{T})$ *be real-valued, and suppose that* $\|\psi\|_\infty \leqq 1$. *Then*

$$\|e^{ir\psi}\|_B \leqq C|r|^2[\log(\|\psi\|_{A(\mathbf{T})} + e^2)]^2,$$

for $|r| \geqq 1$.

PROOF. Applying Lemma 2.7 to the function $e^{ir\psi}$, we see that

$$\|e^{ir\psi}\|_B \leqq \frac{C}{(1-\alpha)^2}[\log(\|e^{ir\psi}\|_{(1-\alpha)} + e^2)]^2, \tag{1}$$

for $0 < \alpha < 1$.

Since $B_{(1-\alpha)}$ is a Banach algebra, it follows that

$$\log(\|e^{ir\psi}\|_{(1-\alpha)} + e^2) \leqq C + |r|\,\|\psi\|_{(1-\alpha)}.$$

By Lemma 2.4 and the assumption that $\|\psi\|_\infty \leqq 1$, we have

$$\|\psi\|_{(1-\alpha)} \leqq C\|\psi\|^{1-\alpha}_{A(\mathbf{T})}.$$

Thus, inequality (1) implies

$$\|e^{ir\psi}\|_B \leqq \frac{C}{(1-\alpha)^2}[C + C|r|\,\|\psi\|^{1-\alpha}_{A(\mathbf{T})}]^2. \tag{2}$$

Choose $1-\alpha = 1/\log(\|\psi\|_{A(\mathbf{T})} + e^2)$. A simple computation in (2) then yields for $|r| \geqq 1$,

$$\| e^{ir\psi} \|_B \leqq C |r|^2 [\log(\|\psi\|_{A(\mathbf{T})} + e^2)]^2,$$

as desired.

We are now in a position to obtain Theorem 2.2. The argument is similar to that of [5]. However, we now apply the estimate of Lemma 2.8 in place of the obvious inequality $\| e^{ir\psi} \|_B \leqq C + C|r| \|\psi\|_{A(\mathbf{T})}$ used in the proof of theorem 1.4 of [5]. Here $|r| \geqq 1$ and ψ is a real-valued function in $A(\mathbf{T})$.

Proof of Theorem 2.2. Fix $|r| \geqq 1$. Let $\psi \in A(\mathbf{T})$ be real-valued and suppose that $\|\psi\|_B \leqq 1$. Let $\psi = \Sigma a_k \psi_k$ be an admissible representation for ψ so that $\Sigma |a_k| \log(\|\psi_k\|_{A(\mathbf{T})} + e^2) < 2$. Since ψ is real-valued, it is easy to see that we may assume $a_k \in \mathbf{R}$ and ψ_k is real-valued, $1 \leqq k \leqq n$. We write

$$\psi = \sum_{\log(\|\psi_k\|_{A(\mathbf{T})} + e^2) \leqq |r|} a_k \psi_k + \sum_{\log(\|\psi_k\|_{A(\mathbf{T})} + e^2) > |r|} a_k \psi_k = \varphi_1 + \varphi_2,$$

and estimate separately $e^{ir\varphi_1}$ and $e^{ir\varphi_2}$.

Consider first the term $e^{ir\varphi_1}$. Notice that φ_1 is real-valued, and $\|\varphi_1\|_\infty \leqq \Sigma |a_k| \leqq \frac{1}{2}\Sigma |a_k| \log(\|\psi_k\|_{A(\mathbf{T})} + e^2) \leqq 1$. Thus, Lemma 2.8 asserts that

$$(1) \qquad \| e^{ir\varphi_1} \|_B \leqq C |r|^2 [\log(\|\varphi_1\|_{A(\mathbf{T})} + e^2)]^2.$$

We now observe that

$$(2) \qquad \begin{aligned} \|\varphi_1\|_{A(\mathbf{T})} &\leqq \sum_{\log(\|\psi_k\|_{A(\mathbf{T})} + e^2) \leqq |r|} |a_k| \, \|\psi_k\|_{A(\mathbf{T})} \\ &\leqq e^{|r|} \sum |a_k| \leqq \tfrac{1}{2} e^{|r|} \sum |a_k| \log(\|\psi_k\|_{A(\mathbf{T})} + e^2) \leqq e^{|r|}. \end{aligned}$$

By (1) and (2), it follows easily that

$$(3) \qquad \| e^{ir\varphi_1} \|_B \leqq C |r|^4.$$

In order to estimate the term $e^{ir\varphi_2}$, we note that for $s \geqq 1$, $\varphi_2^s = \Sigma_{k_1 \in J} \cdots \Sigma_{k_s \in J} a_{k_1} \cdots a_{k_s} \psi_{k_1} \cdots \psi_{k_s}$ is an admissible representation for φ_2^s. Here $J = \{k \mid \log(\|\psi_k\|_{A(\mathbf{T})} + e^2) > |r|\}$. Therefore,

$$(4) \qquad \begin{aligned} \|\varphi_2^s\|_B &\leqq \sum_{k_1 \in J} \cdots \sum_{k_s \in J} |a_{k_1}| \cdots |a_{k_s}| \log(\|\psi_{k_1} \cdots \psi_{k_s}\|_{A(\mathbf{T})} + e^2) \\ &\leqq \sum_{k_1 \in J} \cdots \sum_{k_s \in J} |a_{k_1}| \cdots |a_{k_s}| \left(\sum_{i=1}^{s} \log(\|\psi_{k_i}\|_{A(\mathbf{T})} + e^2) \right) \\ &\leqq \frac{s}{|r|^{s-1}} \sum_{k_1 \in J} \cdots \sum_{k_s \in J} |a_{k_1}| \cdots |a_{k_s}| \prod_{i=1}^{s} \log(\|\psi_{k_i}\|_{A(\mathbf{T})} + e^2) \\ &\leqq \frac{s 2^s}{|r|^{s-1}}. \end{aligned}$$

The penultimate inequality follows since $\log(\|\psi_{k_i}\|_{A(\mathbf{T})}+e^2)>|r|$ for all $k_i\in J$. (Here we use the simple fact that $x_i \geqq N>0$, $1\leqq i\leqq s$, implies that $\Sigma_{i=1}^s x_i \leqq (s/N^{s-1})\Pi_{i=1}^s x_i$.) The last inequality holds since $\Sigma|a_k|\log(\|\psi_k\|_{A(\mathbf{T})}+e^2)<2$.

Using the Taylor series expansion for $e^{ir\varphi_2}$, we obtain by (4)

$$\|e^{ir\varphi_2}\|_B \leqq \|1\|_B+\sum_{s=1}^{\infty}\frac{|r|^s\|\varphi_2^s\|_B}{s!}$$

(5)
$$\leqq C+\sum_{s=1}^{\infty}\frac{|r|^s}{s!}\frac{s2^s}{|r|^{s-1}}\leqq C|r|.$$

Combining (3) and (5) yields

(6)
$$\|e^{ir\psi}\|_B=\|e^{ir(\varphi_1+\varphi_2)}\|_B\leqq\|e^{ir\varphi_1}\|_B\,\|e^{ir\varphi_2}\|_B\leqq C|r|^5.$$

It is well-known that the estimate (6) implies that C^6 functions operate on B (see, for example, [4], chapter 6, section 7). The proof is complete.

The arguments of 2.5–2.8 indicate a definite relationship between the spaces $(B^0,B^1)_f$ and the real interpolation spaces of Lions and Peetre. It is thus not surprising that $(B^0,B^1)_f$ often enjoys an interpolation property. Specifically, we have the following result.

Proposition 2.9. *Let (B^0,B^1) and (D^0,D^1) be two pairs of Banach spaces satisfying the notations of 2.1. Let f be as in 2.1, and suppose that there exists a function $h:[0,\infty)\to[0,\infty)$ so that $f(st)\leqq h(s)f(t)$, for every $s,t\geqq 0$. Let $T: B^0\to D^0$ be a linear operator so that*

$$\|T(x)\|_{D^j}\leqq M_j\|x\|_{B^j},$$

$j=0,1$, for all $x\in B^0$. Then

$$\|T(x)\|_{D_f}\leqq M_1 h\left(\frac{M_0}{M_1}\right)\|x\|_{B_f},$$

for all $x\in B^0$.

The proof of this result is similar in spirit to proof of Theorem 3.2.23 of [1]. The argument is thus left to the reader.

Finally, we note that Theorem 2.2 leads to a generalization of itself.

Theorem 2.10. *Let $f_0(t)=\log(t+e^2)$, and let $B=(A(\mathbf{T}),C(\mathbf{T}))_{f_0}$. Define $D=(B,C(\mathbf{T}))_{f_0}$. Then D is a strongly homogeneous Banach algebra satisfying*

(a) $A(\mathbf{T})\subsetneqq D\subsetneqq C(\mathbf{T})$.

(b) $\sup_{\substack{\psi\in D\\ \psi \text{ real}\\ \|\psi\|_D\leqq 1}}\|e^{ir\psi}\|_D\leqq C|r|^2$, *for* $|r|\geqq 1$.

In particular, C^3 functions operate on D.

PROOF. The arguments of [5] show that D is a strongly homogeneous Banach algebra. Since $B \neq C(\mathbf{T})$ and B is dense in $C(\mathbf{T})$, the norms of B^* and $C(\mathbf{T})^*$ are not equivalent. The discussion following 2.1 thus implies part (a).

Part (b) follows by a proof similar to that given in Theorem 2.2. Let ψ be a real-valued function in $A(\mathbf{T})$ with $\|\psi\|_D \leqq 1$. Let $\psi = \sum a_k\psi_k$ be an admissible representation for ψ so that for all k, $a_k \in \mathbf{R}$, ψ_k is real-valued, and so that $\sum |a_k| \log(\|\psi_k\|_B + e^2) < 2$. For $|r| \geqq 1$, we write

$$\psi = \sum_{\log(\|\psi_k\|_B + e^2) \leqq |r|} a_k\psi_k + \sum_{\log(\|\psi_k\|_B + e^2) > |r|} a_k\psi_k = \varphi_1 + \varphi_2.$$

We now need only replace inequality (1) in the proof of 2.2 by the estimate

$$\|e^{ir\varphi_1}\|_D \leqq \log(\|e^{ir\varphi_1}\|_B + e^2)$$
$$\leqq \log(C|r|^5\|\varphi_1\|_B^5 + C) \leqq C|r|,$$

which itself is a consequence of Theorem 2.2. The remainder of the proof follows by the argument of 2.2.

We conclude by observing that the appropriate analogues of our principal results, Theorems 2.2 and 2.10, remain valid on infinite LCA groups.

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